

TDC (CBCS) Even Semester Exam., 2022

MATHEMATICS

(Honours)

(2nd Semester)

Course No. : MTMHCC-201T

(Real Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks
for the questions

SECTION—A

Answer any ten questions :

2×10=20

1. If M is a neighbourhood of a point x and $N \supset M$, then show that N is also a neighbourhood of x .

2. Show that $\mathbf{N} \times \mathbf{N}$ is countable.

3. Let

$$S = \left\{ 1 + \frac{(-1)^n}{n}, n \in \mathbf{N} \right\}$$

Find supremum and infimum of S .

4. If x is a limit point of a set $S \subset \mathbf{R}$, is x a limit point of $S - \{x\}$? Justify your answer.

5. Define open and closed sets.

6. Obtain the derived set of the following sets :

(i) $\{1, 2, 3, 4, 5\}$

(ii) $\{1, 2, 3, 4, \dots, 500\}$

7. Find the limit of the sequence $\{x_n\}$ if

$$x_n = \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3}$$

8. Give examples of divergent sequences $\{x_n\}$ and $\{y_n\}$ such that $\{x_n + y_n\}$ converges.

9. Prove that the limit of a convergent sequence is unique.

10. Define subsequence and give example.

11. Show that every convergent sequence is a Cauchy sequence.

12. State Bolzano-Weierstrass theorem for sequence.

(Turn Over)

13. Prove that if a series converges, its n th term must necessarily approach to zero.
14. Define alternating series and state when an alternating series is said to be convergent.
15. Find n th partial sum of the series

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots \infty$$

SECTION—B

Answer any five questions : 10×5=50

16. (a) Prove that every open interval in \mathbb{R} is a neighbourhood of each of its points. 2
- (b) Show that the countable union of countable sets is countable. 4
- (c) State and prove Archimedean property. 4
17. (a) Show that every superset of an uncountable set is uncountable. 2
- (b) Let A and B be two non-empty subsets of \mathbb{R} and let
- $$C = \{x + y \mid x \in A \text{ and } y \in B\}$$
- Show that $\sup C = \sup A + \sup B$. 4
- (c) Prove that the set of rational numbers is not order complete. 4

18. (a) If A and B are sets of real numbers, then show that

$$(A \cup B)' = A' \cap B'$$

- (b) Show that union of two open sets is an open set.
- (c) Show that a set is closed if and only if it contains all its limit points.
19. (a) If A and B are sets of real numbers and $A \subset B$, then show that $A' \subset B'$.

- (b) Prove that a set is closed if and only if its complement is open.
- (c) State and prove Bolzano-Weierstrass theorem.

20. (a) Prove that the sequence with n th term

$$x_n = \frac{2n-7}{3n+2}$$

is monotonically increasing and bounded.

- (b) If the sequences $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively, then show that the sequence $\{x_n y_n\}$ is convergent and converges to xy .
- (c) Show that the sequence $\{x_n\}$, where $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n} \quad \forall n \geq 1$ is convergent.

21. (a) Prove that every monotonically increasing sequence bounded above is convergent and converges to its supremum. 3

(b) Prove that the sequence

$$\left\{ \left(1 + \frac{1}{n} \right)^n \right\}$$

is convergent and find its limit. 4

(c) Show that the sequence $\{x_n\}$ is convergent where

$$x_n = 2 + (-1)^n \frac{1}{n}$$

22. (a) Find subsequences of the sequence

$$\left\{ \frac{n+1}{n+2} \right\}$$

(b) Show that the sequence $\{x_n\}$ defined by

$$x_n = \frac{n}{n+1}$$

is a Cauchy sequence. 3

(c) State and prove Cauchy's general principle of convergence for sequence. 4

(Turn Over)

23. (a) Prove that the sequence $\{x_n\}$ converges to the limit l if and only if every sub-sequence of $\{x_n\}$ converges to l . 4

(b) Prove that every Cauchy sequence is bounded. 3

(c) Using Cauchy's general principle of convergence, show that the sequence $\{x_n\}$ where

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

is not convergent. 3

24. (a) Test the convergence of the following series : 3+3=6

(i) $\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \dots \infty$

(ii) $\left(\frac{1}{3}\right)^2 + \left(\frac{1 \cdot 2}{3 \cdot 5}\right)^2 + \left(\frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7}\right)^2 + \dots$

(b) Show that the series

$$\sum (-1)^{n-1} (\sqrt{n^2+1} - n)$$

is conditionally convergent. 3

25. (a) State and prove ratio test. 4

(b) Test the convergence of the following series :

$$\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \dots \quad 3$$

(c) Discuss the convergence of the series

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots \quad 3$$
