PG–377 PGDMAT–11/ MMS–15

P.G. DIPLOMA IN MATHEMATICS
EXAMINATION – JUNE 2019.

ALGEBRA

Time : 3 hours Maximum marks : 75

PART A — (5 × 5 = 25 marks)

Answer any FIVE questions.

1. Prove that a non-empty subset H of the group G is a subgroup of G if and only if

(a) $a, b \in H$ implies that $\dot{a}\dot{b} \in H$.

(b) $a \in H$ implies that $a^{-1} \in H$.

2. If G is a finite group and $a \in G$, then prove that $O(a) / O(G)$.

3. If R is a ring, then prove that for all $a, b \in R$

(a) $AO = OA = 0$.

(b) $a(-b) = (-a)b = -(ab)$.

4. If R is a commutative ring with unit element and M is an ideal of R, then prove that M is a maximal ideal of R if $\frac{R}{M}$ is a field.

5. If V is the internal direct sum of $U_1, U_2, .., U_n$, then prove that V is isomorphic to the external direct sum of $U_1, U_2, .., U_n$ where $U_1, U_2, .., U_n$ are subspaces of the vector space V.

6. If V is a finite dimensional inner product space and W is a subspace of V, then prove that $W^{\perp})^{\perp} = W$.

7. If $f(x) \in F[x]$ is of degree $n \geq 1$, then prove that there is an extension E of F of degree atmost $n!$ in which $f(x)$ has n-roots.

8. If $T \in A(V)$ and $S \in A(V)$ is regular, then prove that STS^{-1} and T have the same minimal polynomial

PART B — (5 × 10 = 50 marks)

Answer any FIVE questions.

9. State and prove Lagrange's theorem.

10. If G is a finite group, then prove that $C_a = O(G)/O(N(a))$.

11. Prove that the ideal $A = (a_0)$ is a maximal ideal of the Euclidean ring *R*. If and only if a_o is a prime element of *R*.

12. If *V* is a finite dimensional vector space and if *W* is a subspace of *V*, then prove that *W* is finite dimensional, $\dim W \leq \dim V$ and $\dim V / W = \dim V - \dim W$.

13. If *V* and *W* are of dimensions *m* and *n* respectively over *F*, then prove that $Hom(V, W)$ is of dimension *mn* over *F*.

14. If *F* is of characteristic '*O*' and if a, b are algebraic over *F*, then prove that there exists an element $C \in F(a, b)$ such that $F(a, b) = F(c)$.

15. (a) Prove that element $\lambda \in F$ is a characteristic root of $T \in A(V)$ if and only if there exists some $v \neq 0$ in *V*, $vT = \lambda v$.

    (b) If $\lambda \in F$ is a characteristic root of $T \in A(V)$, then prove that λ is a root of the minimal polynomial of *T*.

16. Prove that there exists a subspace *W* of *V* invariant under *T* such that $V = V_1 \oplus W$.